Wright 1997, 'On the Philosophical Significance of Frege's Theorem'

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I. The Question

Hume's Principle (HP): For any concepts F and G, the number of Fs is the same as the number of Gs iff F and G can be put in one-one correspondence.

The RHS is a second-order definable property

Frege's Theorem: The axioms of arithmetic can be derived within second-order logic with HP as the sole additional axiom.

"The Dedekind-Peano Famous Five can be reduced to One." (202-3)

And the resulting theory SOL+HP is consistent if SOL is.

The Question: What is the philosophical significance of Frege's Theorem? Does it vindicate a kind of logicism?

II. First Pass (on 'yes' and 'no')

Yes: HP was introduced as a *contextual definition* of 'number.' Caesar Problem is (at worst) one of incompleteness, in which case HP should still be a *partial* definition. And that means that, whatever the complete definition turns out to be, the laws of arithmetic are consequences of it.

No: Consider:

$$Nx : Fx = Nx : Gx \quad \text{(a)}$$

$$\Rightarrow \exists y : y = Nx : Gx \quad \text{(b)}$$

Dilemma: either HP defines (a) (in terms of the existence of a correspondence) or not. If yes, then we can't existentially generalize, since (b) but not (a) requires a nonempty domain. If no, then (a) has content which cannot simply be *stipulated* to be true!

General question: what is the status of *abstraction principles*?

Principles that introduce a unary term-forming operator on familiar sorts of expressions by fixing truth conditions for identity statements between the terms via some equivalence relation among the expressions. (206)

Frege on "directions": "Thus we replace the symbol || by the more generic symbol =, through removing what is specific in the content of the former and dividing it between a and b. We carve up the content in a new way different from the original way, and this yields us a new concept." (207)

> It is a false dilemma: "we have the option, by laying down the Direction abstraction, of *re-conceptualizing*, as it were, the type of state of affairs which is described" by $a||b$. "It is in no sense a further substantial claim that their directions exist and are identical under the described circumstances. But nor is it he case that, by stipulating that the principle is to hold, we thereby forfeit the right to a face-value construal of its LHS..." (208)

Basic worry: Logicism is obviously false! It is neither a logical truth, nor one that can be made true by stipulation, that there are infinitely many objects!

Basic response: This *can* be made true stipulatively, depending on how we conceptualize *objects*. HP only gives us infinitely many new objects because it makes us re-think what it takes for there to be objects!

Cf. Wright's discussion in the earlier article of the syntactic criteria of object hood.

Wright: grant that this picture of logicism could work in principle. Question: does it work for HP?

III. Bad Company

(i) *Bad Company Objection:* The strategy of getting analytic truths from abstraction principles (of which HP is an instance) is bunk, since very similar principles (e.g. Basic Law V) lead to paradox.

Response: Abstractions are attempts to introduce new concepts. It is no problem for the project as a whole if *sometimes* the attempt misfires (213).

(ii) If $R(F,G)$ is an equivalence relation, then so is $P \vee R(F,G)$ for any P whatsoever. Using this an an abstraction over R, we can derive $\neg P \supset \forall F \forall G : \Sigma F = \Sigma G \leftrightarrow R(F,G)$. But then letting the abstraction be a paradoxical one, we can prove P!

Response: You can only do abstraction on real (as opposed to Cambridge) properties of F and G.

(iii) Re-run this with $(\phi F \wedge \phi G) \vee \forall x(Fx \leftrightarrow Gx)$, where co-extensiveness implies that $\phi F \leftrightarrow \phi G$, e.g. 'is finite or has a finite complement'.

General response: Surely it is an ongoing project to try to draw the line between legitimate and illegitimate abstractions. But the fact that there are a few "bad eggs" does not mean we can't use the method until we do so. If we took this sort of "bad company" objection seriously, it would rule out our ability to fix the truth-conditions of sentences by stipulation, or similarly stipulate satisfaction-conditions for properties.

Example: 'P' is true iff the last sentence in Wright's notebook is.

Example: heterological.

IV. Nuisances

Abstractions work by correlating objects with cells in a partition of the universe. Basic Law V goes haywire, in part, because it violates Cantor's Theorem. HP blows up the universe to countable infinitude, but is stable thereafter. But what's to stop there from being (non-Cambridge) equivalence relations that do the opposite?

$\Delta(F,G)$ iff finitely many objects are $Fx \wedge \neg Gx$ or $Gx \wedge \neg Fx$.

Nuisance Principle (NP): $\forall F \forall G : (n(F) = n(G) \leftrightarrow \Delta(F,G))$

Problem: NP entails that the domain is finite, so is inconsistent with HP. Moreover, it looks to have the exact same sort of status with regard to the legitimacy of it's "reconceptualization": hence neither NP nor HP can be analytic.

Response:

NP is not *conservative* – when added to a theory T, it allows one to prove further things *about concepts appearing only in T*. In particular, it allows you to prove that all such concepts are only finitely instantiated, e.g. *sometime, someplace zebra*.

In contrast, HP's constraint on the cardinality only apply to the concepts it introduces. And even that construal is misleading:

It introduces infinitely many numbers, but the numbers are off doing their own thing, away from the original objects.

> One should not confuse its being a logical truth that the truth condition assigned to HP to $Nx : x \neq x = Nx : x \neq x$ is met with the idea that HP all of itself, as it were, settles the existence of $Nx : x \neq x$. A legitimate abstraction, in short, ought to do no more than introduce a concept by fixing truth conditions for statements concerning instances of that concept. In a limiting case, it may associate such statements with truth conditions of such a kind that they are necessarily fulfilled... But that should not blow away the distinction between concept-formation on the one hand and mere axiomatic stipulation about existence on the other. (231)

If Hartry Field is right that classical mathematics is conservative, then HP is as well.

Question: How could it be conservative? Say T originally had a finite domain, including object *a*. Here's a new thing you can prove about *a* given HP: *a* is co-instantiated with infinitely many objects.

Let $F \approx G$ be true iff the Fs can be put in one-one correspondence with the Gs.

Define Ia to be true iff $\exists F : [Fa \wedge F \approx (\lambda x(Fx \wedge x \neq a))]$

Ia iff a falls under an infinite concept.