Frege Against the Formalists (The *Grundgesetze*, and Frege-Hilbert Correspondence)

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Thomae and Heine

Formal vs. Contentual Arithmetic

Contentual Arithmetic: (Frege, Cantor) Numerals and mathematical signs *refer* to numbers and mathematical objects; thus the language of mathematics expresses thoughts, like English does.

Formal Arithmetic: Either explicitly denies that numerals refer to anything (Heine), or says that does not matter whether they do, and that the important characteristics of numerals are fully characterized by their inferential properties (Thomae).

Motivation:

Removes metaphysical and epistemological questions. Helps secure the foundations of mathematics.

Heine:

> I do not answer the question, what is a number, by defining number conceptually, still less by introducing the irrationals as limits whose *existence* would be a presupposition... I take a purely formal point of view by *calling certain tangible signs numbers*, so that the existence of these numbers is not in question. (97)

Thomae:

> The formal conception of the numbers works within more modest limits than the logical. It asks not, what are numbers and what do they demand, but rather it asks, what do we require of the numbers in arithmetic. Now, for the formal conception, arithmetic is a game with signs which one may well call empty, thereby conveying that (in the calculating game) they do not have any content except that which is attributed to them with respect to their behavior under certain combinatorial rules (the game rule). A chess player makes use of his pieces in a similar fashion... (97)

Frege's Objections

1. **Arithmetic becomes a mere *game*, not a science.**

Analogy with chess is dangerous!

Thomae: "The rules of chess are arbitrary; the system of rules in arithmetic is such that, by means of simple axioms, the numbers can be related to intuitive manifolds and, consequently, can perform an essential service for us in the knowledge of nature" (97-8).

Frege: Clearly arithmetic *does* help with such explanation. The problem is that the formal arithmetician is left unable to explain *how* this happens.

The only reason arithmetic is explanatory is (according to Frege) that its language expresses *propositions* ('thoughts') (99-100). But on the Fregean picture, if you remove the referents of the basic lexical items (like numerals), the compositional semantics can't get going. So *formal* arithmetic doesn't express thoughts, and our story for how it can explain nature is removed.

"How could an equation which expressed nothing, which was nothing but a group of figures, be applied?" (100)

Worries:

(1) What, exactly, is this Fregean story? We take one nut, and another, and drop them into a bowl. There are now two nuts in the bowl. Explanation? $1 + 1 = 2$.

So: there's an entity outside of space and time, which '$1 + 1 = 2$' "expresses"; and some sort of isomorphism between the abstract entity and the nuts ensures that there will be two in the bowl. Is that helpful?

Echoing Vann's worries about "Propositionalism"

(2) What about a Carnapian explanation? The doctrine of linguistic frameworks looks a lot like formalism: you set up your system, however you like, and then run with it. There's a *pragmatic* question about which systems you choose to develop; we choose *this* formal game precisely because it happens to be useful.

Sufficient? Underdetermination worries?

Thus, Frege concludes, arithmetic loses its **applicability**; and "it is applicability alone which elevates arithmetic above a game to the rank of a science" (100).

Worry: Is that right?

Applicability is how it gets *funding* as a science; but that's a separate question!

E.g. developments of pure set theory, or modal/epistemic logic. On a commonsense reading of "applicability" these have none; so on this reading, according to Frege, they are not sciences.

Might Frege respond that they *are* applicable, because '$\Box$', 'K', '{' and '}' have *intended interpretations*? They are about things, and therefore *are* applicable.

(1) Too weak? Formal arithmetic is *about* the numerals!

(2) Perhaps the real "applicability puzzle" comes from ***empirical*** **applicability.**

I have no real puzzle with "How could 'K' explain knowledge so well?" We've decided to read it as such precisely because of this.

Then: even if empirical applicability is not *all* that distinguishes arithmetic as a science (so Frege claimed too much), it is one very important feature that it has. Insofar as the formalist has a worse explanation of this than the realist, that tells in favor of the latter.

2. **Arbitrariness worry** (111)

We look to cases where it feels like formal arithmetic supplies us with very little of the desired explanation.

E.g. take '$3+5$'; I am allowed to write '$= 8$' on the end of this formula. Two questions: (i) *why* can I do this, and (ii) how could it be *informative*?

Contentual:

(i) '3', '5', and '8' have objects as referents, and '+' refers to a function (unsaturated object) that, when given 3 and 5 as inputs, outputs 8. Since this is the same object that is referred to by '8', the equality holds.

(ii) Although '3+5' and '8' refer to the same object, they have different senses.

Formal:

(i) Basically, we've just *decided* to use the signs that way.

(ii) The only new knowledge is the sort I gain when I learn that such-and-such is a possible configuration on a chessboard that one can reach from the starting positions in accordance with the rules.

"In formal arithmetic, the rules are independent of a sense. The goal of knowledge does not determine their content; rather, they are laid down arbitrarily" (112). And yet you might think: it certainly doesn't *feel* arbitrary!

Another gloss on arbitrariness: why do we ever feel the need to introduce new number-signs at all? (112)

3. **Formalist fails to meet his original motivations**

(i) Only pushes the problems back!

The only way to explain the applicability of arithmetic would be to supply its symbols with senses. As Frege sees it, all the formalist has done is say "For the sake of developing *pure* mathematics, we don't need to worry about this." But then it seems that the burden of saying how number-signs correspond to the world is just pushed off to the *applied* mathematician (101).

Now whenever we apply arithmetic to a new area, we need a new story. Better to give a completely general one at the beginning.

(ii) Fails to aid in securing the foundations of mathematics (97).

The goal was to show that arithmetic was purely logical, and didn't rest on geometrical intuitions (like length ratios).

Yet in making mathematics about *numerals*, one has made it radically contingent [metaphysical problem], and also made our knowledge of it depend on sense-perception [epistemological problem].

How is this *better* than geometrical intuition?

4. **Formal arithmetician fails to carry through his program consistently**

(i) Can't do justice to mathematical practice.

E.g. showing why $(3+2)-2=3$:

> We want to say something like, "By definition, $(3+2)-2$ is the number which, increased by 2, yields $(3+2)$. This number is 3; therefore $(3+2)$ and 3 coincide" (122).

But this relies on assumptions that are illicit for formal arithmeticians: (a) single-valuedness of subtraction; and (b) relatedly, the use of definite articles ("*the* number which...") and demonstratives ("*This* number...").

Subtracting 2 from a number standardly gives you *one* number. But this seemingly doesn't hold in formal arithmetic, for there is no one number standing behind the symbols '$5-2$', '$(5+2-2)-2$', etc.

Thus it looks like we will either have to radically *change* our mathematical practice, or radically *reinterpret* it.

Thomae gets caught talking in these ways without every offering such a reinterpretation (126-7).

(ii) Trying to construct the irrationals: problems with infinity.

If you're going to make numbers tangible signs, on the face of it there will no longer be infinitely many of them!

Thomae tries to sidestep: "A sequence of [...] numbers [...] is called an infinite sequence if no term in it is last but rather, if according to an instruction to be given, more and more new terms can always be formed." (129)

Frege: If 'can' is human ability, then no series is infinite. If 'can' is god's ability, then every series is.

Analogy with houses; we're going to run out of wood (ink) at some point!

Objection: Surely the modality from 'can' is to be understood as "possible, in view of what the *rule* for constructing the series says." In that case, not even god could continue adding numbers to some series; and conversely, it doesn't matter if we have human limitations.

Frege: "In order to introduce irrationals we require infinitely many numbers, and formal arithmetic has only a finite collection of number-figures. No amount of definition, no amount of twisting and turning can make it otherwise." (134)

"What will be the result? A series that starts with a figure and ends with a figure. Now, one can no doubt provide a definition according to which this written sequence is nevertheless infinite; but what is the point? The infinity required for the irrational is s not achieved in this way. What is the use of the word 'infinite' to us if the thing that matters is missing!" (134)

Objection: This is not obvious. In mathematical practice people *talk about* infinities, but they (of course) never write them down. So why can't the formal arithmetician simply point to an instance of such a proof and say "My rules for infinite series allows *that* kind of reasoning."?

(iii) Trouble with laying down rules for the system

Thomae tries to do this with things like '$a+b=b+a$'; but focus on '$2+1=2+1$'.

> Frege: "This is a surprise. What would someone say who asked for the rules of chess and instead of any answer was shown a group of chess pieces on the chess board?" (113)
>
> That is, if in the process of laying down the rules we hold onto the formalist doctrine, then these are just uninterpreted symbols – it's hard to see how they could be

"We must first have rules for the manipulation of the figures in the game itself, and these can be laid down completely arbitrarily without regard for any sense. Second, we must then have rules for how we are to manage these same figures as signs in the theory of the game; and these cannot be arbitrary but must be guided by the senses which these signs express, by means of their arrangement, in the theory of the game" (116).

informative! So it looks like we can't hold onto the formalist picture while laying down the *rules* of the game; instead, the symbols in these rules must have referents. Which referents? They better not be the one's in contextual arithmetic!! (114)

The worry, of course, is that however you spell out the semantics for the *theory* of arithmetic will require you to do all you would have needed to do in just giving for arithmetic proper, like contextual arithmetic does.

Summing up: "Indeed, we see the formal arithmetician breaks character again. The formal conception is a shield that is held up as long as questions concerning the reference of the signs threaten. When the danger passes, it is dropped; for it is in the end a burden, after all" (127).

Frege-Hilbert Correspondence

Frege on Formalization

One should not build a formalism and *then* look for an area to apply it. Rather, one should try to investigate an area, and be mindful of when natural language is clunky, which will show you where to introduce formalisms.

Worry about it becoming mechanical? (33) **Lignification:** "When a tree lives and grows it must be soft and succulent. But if what was succulent did not in time turn into wood, the tree could not reach a significant height. On the other hand, when all that was green has turned into wood, the tree ceases to grow." (33)

The picture: we selectively implement formalism which makes certain (well-worked-out) processes automatic/thoughtless, so that we can direct our attention elsewhere. "What was originally saturated with thoughts hardens in time into a mechanism which partly relieves the scientist from having to think." (33)

Sorensen's magical pencil.

Controversy 1: Definitions and Axiomatizations

Frege:

There should be a clean divide between *definitions* and *axioms/theorems*. A definition lays down how a term is to be used; an axiom is an obvious truth that only uses terms that have already been fully defined.

E.g. Looking at Hilbert's axiom: "If A, B, and C are points on a line and B lies between A and C, then B also lies between C and A."

Frege: this better only be introduced after we have defined what 'points on a line' and 'lies between' mean! (37).

Hilbert:

Basically a holist (or Carnapian?) about theoretical terms:

> To try to give a definition of a point in three lines [of text] is to my mind an impossibility, for only the whole structure of axioms yields a complete definition. Every axiom contributes something... (40)
>
> If one is looking for other definitions of 'point', e.g. through paraphrase in terms of extensionless etc., then I must indeed oppose such attempts in the most decisive way; one is looking for something one can never find because there is nothing there... (39)

Something seems right about this. Frege's conception of definition seems to require a very *foundationalist epistemology*. Yet theoretical terms often can only be understood in terms of their role within the larger theory.

E.g. 'electron' or 'set.' Must I give a full, rigorous definition of what a set is *before* I give you the axioms?

Frege:

Giving definitions *only* by laying down axioms is hazardous – what if the axioms fail to fully characterize the term, i.e. fail to determine a complete extension? Example of an axiomatization of "congruent" which fails to say whether $2 \equiv 8 (mod\ 3)$

"Your system of definitions is like a system of equations with several unknowns, where there remains a doubt whether the equations are soluble and, especially, whether the unknown quantities are uniquely determined" (45).

This worry certainly seems to have teeth given the realist conception of mathematics that Frege is coming from.

Hilbert:

Basically, he's not a realist in the way Frege is. We're not trying to fully characterize something out in the world; we're just making a system that has certain properties.

E.g. we could apply his axiomatization of geometry to a domain of chimney-sweeps.

Controversy 2: Existential Commitment

Frege:

Our knowledge that axioms are true flows from their self-evidence. Their truth guarantees that they don't contradict one another (37).

Hilbert:

Reverse that! "If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence." (39-40)

Frege:

"Huh??" The only way he can see to show that two requirements don't contradict one another is to point to something that *exists* and has both (43). Yet Hilbert seems to think that he can get existence from pure non-contradiction, or even from an axiom ("there are at least two points on a line") (45).

Consider:

Explanation. We imagine objects we call Gods.

Axiom 1. All Gods are omnipotent.

Axiom 2. All Gods are omnipresent.

Axiom 3. There is at least one God.

Surely showing that these are consistent would not suffice to prove that there is an omnipotent, omnipresent God!

E.g. applying Carnap's linguistic-framework doctrine to religious disputes

Hilbert, as far as I can tell, gives no direct reply to this worry.

Would he refuse to apply this sort of criterion outside of mathematics? Perhaps it only makes sense in domains that we can be seen as *constructing*, in some sense?