The Gambler's Fallacy

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I. The Gambler's fallacy

The **gambler's fallacy** (aka the "law of small numbers") is the tendency to think that short sequences of random events will have tend to have similar statistical properties to long sequences.

Example: A **koin** is a bit of code on a computer that, when 'flipped', can land either 'heads' or 'tails'. You know that on long sequences, koins tend to land heads around 50% of the time.

The gambler's fallacy: thinking that sequences like TTTH are more likely than those like TTTT.
→ As a result, if you see it land TTT, you think H is more likely than T on the next toss.

The former is "more representative" of the "around-50% heads" tendency.

I.e. $cr(TTTH|TTT) > 0.5$.

Why think it's a fallacy?

- A *fair* koin—one whose tosses are statistically independent—will tend to land 50% heads on long sequences.
- So if all you know is that koins tend to land heads on 50% of long sequences, you have no reason to think it's not fair.
- Therefore, upon seeing it land T, you should not expect it to be more likely to land H than T on the next toss.

I.e. $cr(H_{i+1}|H_i) = cr(H_{i+1})$.

Claim: This reasoning is mistaken—the GF is not a fallacy.
On natural precisifications of the claim "it tends to land heads 50% of the time", knowing this *does* give you reason to expect that short sequences will share this property, i.e. that if it lands T, it's more likely to land H on the next toss.

I.e. $cr(H_{i+1}|T_i) > 0.5$.

II. A simple case

Suppose you know that the koin is either Steady, Switchy, or Sticky:

Hypotheses about objective chances.

Steady: If it just landed heads (tails), it's 50% likely to land heads (tails) on the next toss.
Switchy: If it just landed heads (tails), it's *40%* likely to land heads (tails) on the next toss.
Sticky: If it just landed heads (tails), it's *60%* likely to land heads (tails) on the next toss.

More likely to "switch" from what it was.

More likely to "stick" to what it was.

All three hypotheses will lead to ≈ 50% heads on long sequences.

Suppose you know it just landed tails (you don't know what it landed before that). Let *cr* be your credence function.

What's your credence it'll land heads next? By total probability:

$$cr(H) = cr(Switchy) \cdot cr(H|Switchy) + cr(Steady) \cdot cr(H|Steady) + cr(Sticky) \cdot cr(H|Sticky)$$
$$= cr(Switchy) \cdot 0.6 + cr(Steady) \cdot 0.5 + cr(Sticky) \cdot 0.4$$

Notice that $cr(H) > 0.5$ iff $cr(Switchy) > cr(Sticky)$.
$\rightarrow$ You should commit the GF iff $cr(Switchy) > cr(Sticky)$!

Q: Is it reasonable to have $cr(Switchy) > cr(Sticky)$?

Yes! Switchy makes what you know about koins—*they usually land heads around 50% of the time* (E)—more likely than Sticky.
$\rightarrow$ So E confirms Switchy.

Why? Let cr_0 be your prior and $cr(\cdot) = cr_0(\cdot|E)$.

Assume cr_0 is $\frac{1}{3}$ in each of *Switchy*, *Sticky* and *Steady*.

Small case:

- Let $E =$ *it landed heads 50% of the time on 2 tosses*. This is equivalent to $TH \vee HT$, i.e. "it landed one way then switched".
- Thus $cr_0(E|Switchy) = 0.6$, while $cr_0(E|Sticky) = 0.4$.
- Meanwhile, $cr_0(E) = 0.5$.
  $\Rightarrow cr_0(E|Switchy) = 0.6 > 0.5 = cr_0(E)$.
  $\Rightarrow$ Since relevance is symmetric, $cr_0(Switchy|E) > cr_0(Switchy)$!
- So $cr(H) = 0.4 \cdot 0.6 + 0.33 \cdot 0.5 + 0.27 \cdot 0.4 \approx 0.513$.
  Since this is greater than 0.5, that's the gambler's fallacy!

Meanwhile, $cr_0(E|Steady) = 0.5$.

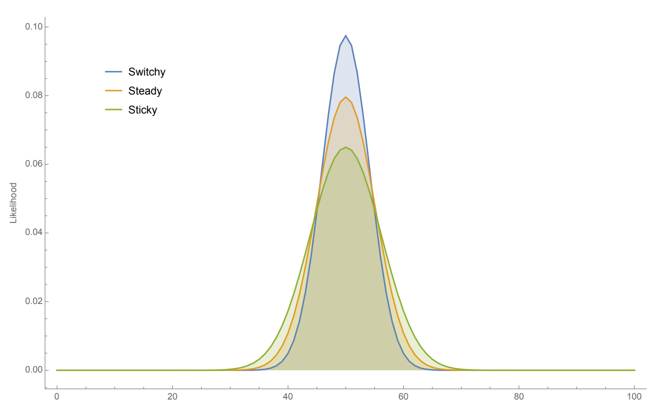
Why?

$cr_0(Switchy|E) = \frac{cr_0(Switchy)cr_0(E|Switchy)}{cr_0(E)}$ $= \frac{1/3 \cdot 0.6}{0.5} = 0.4$. Meanwhile, $cr_0(Steady|E) = \frac{1}{3} \approx 0.33$ and $cr_0(Sticky|E) = \frac{4}{15} \approx 0.27$.

General case:

In general, for *any* number of tosses n, $E =$ *The coin landed heads around 50% of the time in n tosses* will be made more likely by *Switchy* than by *Sticky*, and therefore will be confirm *Switchy*.

E.g. the likelihoods of various numbers of heads on 100 tosses:



Since the bell curve is tighter on the Switchy hypothesis, that means $cr_0(E|Switchy) > cr_0(E|Sticky)$. So the reasoning generalizes.